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Structure of atypical representations of the Lie superalgebras sl(m/n)

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Abstract. In contrast to Lie algebras, Lie superalgebras contain typical and atypical irreducible representations. Character formulae for typical representations of classical Lie superalgebras have been known for a long time, but the structure of atypical representations remained unsolved. In this paper we construct the character formula for atypical representations of the special linear Lie superalgebras sl(m/n).

1. Introduction

Since the work of Corwin *et al* (1975), Lie superalgebras have become increasingly important in theoretical physics. We only mention their evidence in supersymmetry (Fayet and Ferrara 1977), supergravity (van Nieuwenhuizen 1981) and nuclear physics (Iachello 1980) here. Simple Lie superalgebras were classified completely (Kac 1977, Scheunert 1979) and it was shown that classical Lie superalgebras can be described by a Cartan matrix or, equivalently, by a Kac–Dynkin diagram.

Finite-dimensional irreducible representations (irreps) of Lie superalgebras were studied by Kac (1978) and categorised as being either typical or atypical. Typical representations have properties analogous to those of finite-dimensional irreps of Lie algebras and are relatively easy to handle. Kac (1978) constructed a character formula for typical irreps of classical Lie superalgebras. Atypical representations are much harder to deal with. Various techniques have been used in order to get a deeper insight into the structure of atypical representations, but still no completely general character formulae exist. Among the techniques used we mention methods based on superfields (Farmer and Jarvis 1983, 1984), tensors and supertableaux (Balantekin and Bars 1981, 1982, King 1983), shift operators (Hughes 1981, Van der Jeugt 1984, 1985a), weight spaces (Hurni and Morel 1982, 1983) and generating functions (Sharp *et al* 1985). Atypical representations are important for several reasons. Firstly, adjoint representations of Lie superalgebras are usually atypical. Secondly, it was shown recently (Van der Jeugt 1985b) that the phenomenon of multiplet shortening in supergravity models (Freedman and Nicolai 1984) is explained by the structure of atypical representations of the underlying superalgebra.

In the present paper, we study finite-dimensional atypical representations of the Lie superalgebras sl(m/n), sometimes denoted by SU(m/n), spl(m, n) or A($m-1$, $n-1$). The main result is the construction of a character formula for atypical representations, given by (4.31) when precisely one of the atypicality conditions is satisfied.

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The outline of the paper is as follows: in § 2, the algebra is defined and the Chevalley basis is constructed. It is also shown how the atypicality conditions are derived. In § 3, a very useful theorem is proven, which shows the existence of a special highest weight vector in the Verma module $\tilde{V}(\Lambda)$, if Λ corresponds to the highest weight of an atypical representation. This property is used in § 4 in order to obtain the character formula for atypical representations of $\mathrm{sl}(m/n)$. Section 5 gives an example for $\mathrm{L}=\mathrm{sl}(3/2)$.

2. Conventions and notation for sl(m/n)

In matrix notation, the standard form of the Lie superalgebra $\mathrm{sl}(m/n)$ is defined as follows (Kac 1977):

$$\mathrm{sl}(m/n)=\left\{x=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\middle|\, \mathrm{str}\, x=\mathrm{Tr}\, a-\mathrm{Tr}\, d=0\right\} \tag{2.1}$$

where a is a complex $(m\times m)$ matrix, b a $(m\times n)$ matrix, c a $(n\times m)$ matrix and d a $(n\times n)$ matrix. The Lie superalgebra $\mathrm{sl}(m/n)$ is simple if $m\neq n$. If $m=n$, it contains the non-trivial ideal $\mathbb{C}\cdot I_{2m}$, and then $\mathrm{sl}(n/n)/\mathbb{C}\cdot I_{2m}$ is simple. In this paper, we shall prove the results only for $m\neq n$, but they can easily be deduced for $m=n$. The even subspace of $\mathrm{sl}(m/n)$ is the space for which b and c are zero and the odd subspace consists of matrices with $a=0$ and $d=0$.

Let e_{ij} be the matrix with entry 1 in the ith row and jth column, and 0 elsewhere. We set

$$I_m=\sum_{i=1}^{m} e_{ii} \tag{2.2}$$

$$I_n=\sum_{i=m+1}^{m+n} e_{ii}. \tag{2.3}$$

The Cartan subalgebra H of $\mathrm{sl}(m/n)$ is spanned by

$$\begin{aligned} &h_i=e_{ii}+(1/n)I_n \qquad (i=1,\ldots,m)\\ &h_{m+i}=(1/m)I_m+e_{m+i,m+i} \qquad (i=1,\ldots,n)\\ &h_0=(1/m)I_m+(1/n)I_n. \end{aligned} \tag{2.4}$$

Only $(m+n-1)$ of the elements (2.4) are independent since

$$\begin{aligned} &\sum_{i=1}^{m} h_i-mh_0=0\\ &\sum_{i=m+1}^{m+n} h_i-nh_0=0. \end{aligned} \tag{2.5}$$

H is contained in the set of diagonal $(m\times n)\times(m+n)$ matrices $\mathrm{D}=\{d=\mathrm{diag}(d_{11},\ldots,d_{m+n,m+n})\}$ and the linear forms ε_i $(i=1,\ldots,m)$ and δ_j $(j=1,\ldots,n)$ are defined as follows:

$$\begin{aligned} &\varepsilon_i(d)=d_{ii}\\ &\delta_j(d)=d_{m+j,m+j}. \end{aligned} \tag{2.6}$$

The same notation is used for the restrictions of ε_i and δ_j to H. Then, the following relations can be checked for $i=0,1,\ldots,m+n$:

$$[h_i, e_{jk}]=(\varepsilon_j-\varepsilon_k)(h_i)e_{jk} \qquad 1\leqslant j, k\leqslant m$$

$$\begin{aligned} &[h_i, e_{m+j,m+k}] = (\delta_j - \delta_k)(h_i) e_{m+j,m+k} && 1 \leqslant j, k \leqslant n \\ &[h_i, e_{j,m+k}] = (\varepsilon_j - \delta_k)(h_i) e_{j,m+k} && 1 \leqslant j \leqslant m, 1 \leqslant k \leqslant n \\ &[h_i, e_{m+j,k}] = (-\varepsilon_k + \delta_j)(h_i) e_{m+j,k} && 1 \leqslant j \leqslant n, 1 \leqslant k \leqslant m. \end{aligned} \tag{2.7}$$

This shows that the even (resp odd) roots are given by

$$\begin{aligned} &\Delta_0 = \{\varepsilon_j - \varepsilon_k (1 \leqslant j, k \leqslant m), \delta_p - \delta_q (1 \leqslant p, q \leqslant n)\} \\ &(\text{resp } \Delta_1 = \{\pm(\varepsilon_j - \delta_k)(1 \leqslant j \leqslant m, 1 \leqslant k \leqslant n)\}). \end{aligned} \tag{2.8}$$

A set of simple roots is given by

$$(\alpha_1, \ldots, \alpha_{m+n-1}) = (\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n) \tag{2.9}$$

where $\alpha_m = \varepsilon_m - \delta_1$ is the only odd simple root. The root vectors E_i and F_i corresponding with α_i and $-\alpha_i$, respectively, are equal to

$$\begin{aligned} &E_i = e_{i,i+1} && F_i = e_{i+1,i} && i = 1, \ldots, m-1 \\ &E_m = e_{m,m+1} && F_m = e_{m+1,m} && \\ &E_{m+i} = e_{m+i,m+i+1} && F_{m+i} = e_{m+i+1,m+i} && i = 1, \ldots, n-1. \end{aligned} \tag{2.10}$$

Then we have

$$[E_i, F_j] = \delta_{ij} H_j \tag{2.11}$$

with

$$\begin{aligned} &H_i = h_i - h_{i+1} && i = 1, \ldots, m-1, m+1, \ldots, m+n-1 \\ &H_m = h_m + h_{m+1} - h_0. && \end{aligned} \tag{2.12}$$

The Lie superalgebra $\mathrm{sl}(m/n)$ is generated by the set $\{H_i, E_i, F_i\}$ $(1 \leqslant i \leqslant m+n-1)$, and the generators satisfy

$$[H_i, E_j] = c_{ij} E_j \qquad [H_i, F_j] = -c_{ij} F_j \tag{2.13}$$

with

$$C = [c_{ij}] = \left(\begin{array}{ccccc:c:ccccc} & & & & & m & & & & m+n-1 & \\ & & & & & \downarrow & & & & \downarrow & \\ 2 & -1 & & & & & & & & & \\ -1 & 2 & -1 & & & & & & & & \\ & -2 & 2 & & & & & & & & \\ & & & \ddots & -1 & & & & & & \\ & & & -1 & 2 & -1 & & & & & \\ \hdashline & & & & -1 & 0 & 1 & & & & \\ \hdashline & & & & & -1 & 2 & -1 & & & \\ & & & & & & -1 & 2 & -1 & & \\ & & & & & & & -1 & \ddots & & \\ & & & & & & & & & 2 & -1 \\ & & & & & & & & & -1 & 2 \end{array}\right) \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \leftarrow m \\ \\ \\ \\ \\ \leftarrow m+n-1. \end{array} \tag{2.14}$$

The non-degenerate supersymmetric bilinear form $(x, y) = \mathrm{str}(xy)$ on $\mathrm{sl}(m/n)$ is still non-degenerate on H, and hence induces a non-degenerate bilinear form (,) on the dual space H*:

$$(\varepsilon_i, \varepsilon_j) = \delta_{ij} - 1/(m-n)$$
$$(\varepsilon_i, \delta_k) = -1/(m-n) \tag{2.15}$$
$$(\delta_k, \delta_l) = -\delta_{kl} - 1/(m-n) \qquad 1 \leq i, j \leq m,\ 1 \leq k, l \leq n$$

where δ_{ij} or δ_{kl} is the Kronecker symbol.

Note that (2.9) determines the positive roots, namely

$$\begin{aligned} \Delta_0^+ &= \{\varepsilon_j - \varepsilon_k (1 \leq j < k \leq m),\ \delta_p - \delta_q (1 \leq p < q \leq n)\} \\ \Delta_1^+ &= \{\varepsilon_j - \delta_k (1 \leq j \leq m,\ 1 \leq k \leq n)\}. \end{aligned} \tag{2.16}$$

Hence, in the notation of Kac (1978)

$$\bar{\Delta}_1^+ = \Delta_1^+. \tag{2.17}$$

As usual, ρ is defined by

$$\rho = \rho_0 - \rho_1 = \tfrac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha - \tfrac{1}{2} \sum_{\beta \in \Delta_1^+} \beta \tag{2.18}$$

which gives explicitly

$$\rho = \tfrac{1}{2} \sum_{i=1}^{m} (m-n-2i+1)\varepsilon_i + \tfrac{1}{2} \sum_{j=1}^{n} (m+n-2j+1)\ \delta_j. \tag{2.19}$$

Furthermore, if $\Lambda \in \mathrm{H}^*$, we set

$$a_i = \Lambda(H_i) \qquad i = 1, \ldots, m+n-1. \tag{2.20}$$

With every $\Lambda \in \mathrm{H}^*$, an irreducible module $V(\Lambda)$ with highest weight Λ is associated as follows. First, the elements $\{H_i, E_i\}$ $(i = 1, \ldots, m+n-1)$ generate a Borel subalgebra B of $\mathrm{sl}(m/n)$. Then, $\langle v_\Lambda \rangle$ is a one-dimensional B module by

$$\begin{aligned} H_i v_\Lambda &= a_i v_\Lambda \\ E_i v_\Lambda &= 0. \end{aligned} \tag{2.21}$$

Following Kac (1978), we set $\tilde{V}(\Lambda) = \mathrm{Ind}_{\mathrm{B}}^{\mathrm{L}} \langle v_\Lambda \rangle$ (with $\mathrm{L} = \mathrm{sl}(m/n)$) and $V(\Lambda) = \tilde{V}(\Lambda)/I(\Lambda)$, where $I(\Lambda)$ is the unique maximal submodule of $\tilde{V}(\Lambda)$. Kac proved that all finite-dimensional irreps of $\mathrm{sl}(m/n)$ are of type $V(\Lambda)$ and that $V(\Lambda)$ is finite-dimensional if and only if

$$a_i \in \mathbb{Z}_+ \qquad \text{for } i \neq m. \tag{2.22}$$

In general, an irreducible highest weight representation $V(\Lambda)$ is typical if

$$(\Lambda + \rho, \beta) \neq 0 \qquad \forall \beta \in \bar{\Delta}_1^+. \tag{2.23}$$

Making use of (2.15), (2.18) and (2.20), we deduce that

$$(\Lambda + \rho, \varepsilon_l - \delta_j) = a_l + a_{l+1} + \ldots + a_m - a_{m+1} - a_{m+2} - \ldots - a_{m+j-1} + m - l - j + 1$$
$$(1 \leq l \leq m,\ 1 \leq j \leq n). \tag{2.24}$$

These expressions give rise to the atypicality conditions for $\mathrm{sl}(m/n) = \mathrm{A}(m-1, n-1)$. For finite-dimensional typical representations, a character formula is well known (Kac 1978). In the following sections, we shall obtain a character formula for atypical irreps.

3. Existence of an eigenvector of B

In the present section we shall prove that if $(\Lambda+\rho, \beta)=0$ for a certain β of $\bar{\Delta}_1^+$, then there exists a weight vector $v_{\Lambda-\beta}$ of $\tilde{V}(\Lambda)$ such that $v_{\Lambda-\beta}$ is an eigenvector of B. The proof is very simple for $\beta=\alpha_m$, the unique odd simple root.

Lemma 1. There exists a weight vector $w_{m,1}$ of $\tilde{V}(\Lambda)$, with corresponding weight $\Lambda-\alpha_m=\Lambda-(\varepsilon_m-\delta_1)$, such that

$$\begin{aligned} &E_i w_{m,1}=0 \qquad \text{for } i\neq m \\ &E_m w_{m,1}=a_m v_\Lambda . \end{aligned} \tag{3.1}$$

Proof. Let $w_{m,1}=F_m v_\Lambda$. This is a weight vector with weight $\Lambda-\alpha_m$, and for $i\neq m$ we have

$$E_i w_{m,1}=F_m E_i v_\Lambda=0$$

whereas for $i=m$

$$E_m w_{m,1}=(-F_m E_m+H_m)v_\Lambda=a_m v_\Lambda .$$

A similar property for the roots $\varepsilon_m-\delta_j$ $(j=2,\ldots,n)$ of $\bar{\Delta}_1^+$ is given in lemma 2. The proof of lemma 2 is more technical, although we only make use of the generating relations (2.11) and (2.13).

Lemma 2. For $j\in\{2,\ldots,n\}$, there exists a weight vector $w_{m,j}$ of $\tilde{V}(\Lambda)$ with weight $\Lambda-\varepsilon_m+\delta_j$, such that

$$\begin{aligned} &E_i w_{m,j}=0 \qquad \forall i\neq m \\ &E_m w_{m,j}=(a_m-a_{m+1}-\ldots-a_{m+j-1}-j+1)w^+_{m,j-1} \end{aligned} \tag{3.2}$$

where $w^+_{m,j-1}$ is a weight vector with weight $\Lambda-\delta_1+\delta_j$.

Proof. First, we check the result for $j=2$. We put

$$x=F_m F_{m+1} v_\Lambda \qquad y=F_{m+1}F_m v_\Lambda . \tag{3.3}$$

Obviously, $E_i x=E_i y=0$ for all $i\neq m, m+1$. For $i=m+1$, we find, making use of (2.11), (2.13) and (2.21),

$$\begin{aligned} &E_{m+1}x=a_{m+1}w_{m,1} \\ &E_{m+1}y=(a_{m+1}+1)w_{m,1} . \end{aligned}$$

Hence we let

$$w_{m,2}=(a_{m+1}+1)x-a_{m+1}y \tag{3.4}$$

for which $E_{m+1}w_{m,2}=0$. Then, we see

$$\begin{aligned} E_m w_{m,2}&=[(a_{m+1}+1)(a_m-1)-a_{m+1}a_m]F_{m+1}v_\Lambda \\ &=(a_m-a_{m+1}-1)F_{m+1}v_\Lambda \end{aligned} \tag{3.5}$$

and $F_{m+1}v_\Lambda$ is a weight vector with weight $\Lambda-\delta_1+\delta_2$. Moreover, we introduce the following notation:

if $\quad w=\sum c_{i_1\ldots i_k}(a_1,\ldots,a_{m+n-2})F_{i_1}F_{i_2}\ldots F_{i_k}v_\Lambda \qquad (\text{all } i_k\leqslant m+n-2)$

then $\quad w^+=\sum c_{i_1\ldots i_k}(a_2,\ldots,a_{m+n-1})F_{i_1+1}\ldots F_{i_k+1}v_\Lambda \qquad (3.6)$

where $c_{i_1\ldots i_k}$ are polynomial functions in $a_1,\ldots,a_{m+n-2}$. Hence, w^+ is deduced from w by increasing all the a_i and F_i indices by 1. With this convention, we see that

$$E_m w_{m,2}=(a_m-a_{m+1}-1)w_{m,1}^+. \qquad (3.7)$$

Lemma 2 is proven in the case $j=2$. For $j>2$, we shall prove the property by means of induction on j. Hence, we may start from the following induction hypothesis: $(2\leqslant j\leqslant n-1)$.

Let

$$w_{m,j}=\sum_{(k)}\lambda_k(a_{m+1},\ldots,a_{m+j-1})P_k(F_m,F_{m+1},\ldots,F_{m+j-1})v_\Lambda. \qquad (3.8)$$

Here, k belongs to a subset S of the set $P=\{\sigma\,|\,\sigma \text{ is a permutation of } (m, m+1,\ldots,m+j-1)\}$. In order to describe S, let $\cong$ be the equivalence relation on P defined by

$$\sigma_1\simeq\sigma_2 \Leftrightarrow F_{\sigma_1(m)}\ldots F_{\sigma_1(m+j-1)}=F_{\sigma_2(m)}\ldots F_{\sigma_2(m+j-1)} \qquad \text{in } \mathrm{U}(\mathrm{sl}(m/n)). \qquad (3.9)$$

Then S is the set of equivalence classes in P for the relation $\cong$; when an equivalence class is represented by a particular representative, S can be seen as a subset of P. Since $[F_p,F_q]=0$ if $q\neq p\pm1$ in $\mathrm{sl}(m/n)$, we see that the cardinality of S equals 2^{j-1} (whereas the cardinality of P is, of course, $j!$). For $k\in S$, $F_{k(m)}F_{k(m+1)}\ldots F_{k(m+j-1)}$ is denoted by $P_k(F_m,F_{m+1},\ldots,F_{m+j-1})$ and $\lambda_k(a_{m+1},\ldots,a_{m+j-1})$ is the corresponding coefficient which is a polynomial in $a_{m+1},\ldots,a_{m+j-1}$. The vector (3.8) satisfies the following properties:

$$\begin{aligned} &E_i w_{m,j}=0 \qquad \forall i\neq m\\ &E_m w_{m,j}=(a_m-a_{m+1}-\ldots-a_{m+j-1}-j+1)w_{m,j-1}^+. \end{aligned} \qquad (3.10)$$

This describes the induction hypothesis completely and it is easy to verify it for $j=2$.

Now we will prove the property for $(j+1)$. For this purpose, define, in a similar way as in (3.3),

$$\begin{aligned} X&=F_m w_{mj}^+\\ &=F_m\sum_k\lambda_k(a_{m+2},\ldots,a_{m+j})P_k(F_{m+1},\ldots,F_{m+j})v_\Lambda \qquad (3.11)\\ Y&=\sum_k\lambda_k(a_{m+2},\ldots,a_{m+j})P_k(F_{m+1},\ldots,F_{m+j})F_m v_\Lambda. \end{aligned}$$

Then, for $i\neq m, m+1$,

$$E_iX=F_mE_iw_{mj}^+.$$

However since $i>m+1$, $E_iw_{mj}^+$ is equal to $(E_{i-1}w_{mj})^+$, which vanishes according to (3.10). Hence

$$E_iX=0 \qquad \text{for } i\neq m, m+1. \qquad (3.12)$$

Making use of the explicit form of X, the action of E_i implies that all F_i are replaced by H_i:

$$0=E_iX=F_m\sum_k\lambda_k(a_{m+2},\ldots,a_{m+j})P_k(F_{m+1},\ldots,H_i,\ldots,F_{m+j})v_\Lambda$$
$$i\neq m, m+1. \qquad (3.13)$$

In (3.13), H_i is substituted by its eigenvalue $(a_i + c)$, where the number c is 0, 1 or 2, if no, one or both of the operators F_{i-1}, F_{i+1} are placed behind H_i in the explicit form of P_k, respectively. But then $(i \neq m, m+1)$

$$E_i Y = \sum_k \lambda_k(a_{m+2}, \ldots, a_{m+j}) P_k(F_{m+1}, \ldots, H_i, \ldots, F_{m+j}) F_m v_\Lambda = 0 \tag{3.14}$$

because H_i is substituted by the same eigenvalue, since $[H_i, F_m] = 0$. Now, consider the case of $E_i = E_{m+1}$. Obviously

$$E_{m+1} X = F_m E_{m+1} w^+_{mj}.$$

The vector $E_{m+1} w^+_{m,j}$ is not equal right away to $(E_m w_{mj})^+$, since the mth row of the Cartan matrix C (2.14) differs from the next rows. But that $E_{m+1} w^+_{m,j}$ is still related to $(E_m w_{mj})^+$ can be seen as follows.

(i) In order to compute $E_{m+1} w^+_{mj}$, all F_{m+1} in w^+_{mj} are replaced by H_{m+1} and then H_{m+1} is substituted by $a_{m+1} - c_{m+1,m+2} = a_{m+1} + 1$ if F_{m+2} appears behind H_{m+1}, and by a_{m+1} otherwise.

(ii) In order to compute $E_m w_{mj}$, all F_m in w_{mj} are replaced by H_m and then H_m is substituted by $a_m - c_{m,m+1} = a_m - 1$ if F_{m+1} appears behind H_m, and by a_m otherwise. Hence, if we take $E_m w_{mj}$, replace all a_m by $-a_m$, then change the signs of all coefficients, and finally increase all indices by 1, we must obtain $E_{m+1} w^+_{mj}$. But $E_m w_{mj}$ is given in (3.10), hence

$$E_{m+1} w^+_{mj} = (a_{m+1} + a_{m+2} + \ldots + a_{m+j} + j - 1) w^{++}_{m,j-1} \tag{3.15}$$

or

$$E_{m+1} X = (a_{m+1} + a_{m+2} + \ldots + a_{m+j} + j - 1) F_m w^{++}_{m,j-1}. \tag{3.16}$$

Then we find

$$E_{m+1} Y = \sum_k \lambda_k(a_{m+2}, \ldots, a_{m+j}) P_k(H_{m+1}, F_{m+2}, \ldots, F_{m+j}) F_m v_\Lambda. \tag{3.17}$$

The expression on the right-hand side of (3.17) is the same as in the one for $E_{m+1}X$, but with F_m placed at the end. This implies that, compared to $E_{m+1}X$, all a_{m+1} are replaced by $a_{m+1} - c_{m+1,m} = a_{m+1} + 1$. This leads to

$$E_{m+1} Y = (a_{m+1} + a_{m+2} + \ldots + a_{m+j} + j) F_m w^{++}_{m,j-1}. \tag{3.18}$$

Hence, if we put

$$w_{m,j+1} = (a_{m+1} + a_{m+2} + \ldots + a_{m+j} + j) X - (a_{m+1} + a_{m+2} + \ldots + a_{m+j} + j - 1) Y \tag{3.19}$$

$w_{m,j+1}$ is precisely of the form (3.8) with $j \to j+1$, and from (3.12), (3.14), (3.16) and (3.18) we see that

$$E_i w_{m,j+1} = 0 \qquad \forall i \neq m. \tag{3.20}$$

Finally, since

$$E_m X = H_m w^+_{mj} = (a_m - 1) w^+_{mj}$$

and

$$E_m Y = \sum_k \lambda_k(a_{m+2}, \ldots, a_{m+j}) P_k(F_{m+1}, \ldots, F_{m+j}) H_m v_\Lambda = a_m w^+_{mj}$$

we find

$$E_m w_{m,j+1} = [(a_{m+1} + a_{m+2} + \ldots + a_{m+j} + j)(a_m - 1) - (a_{m+1} + \ldots + a_{m+j} + j - 1)a_m] w^+_{m,j}$$
$$= (a_m - a_{m+1} - \ldots - a_{m+j} - j) w^+_{mj} \qquad (3.21)$$

and obviously w^+_{mj} is a weight vector with weight $\Lambda - \delta_1 + \delta_{j+1}$.

In the next lemma, the property (3.2) will be extended for the case of weight vectors with weight $\Lambda - (\varepsilon_l - \delta_l)$ with $1 \leqslant l \leqslant m-1$.

Lemma 3. For $l \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, there exists a weight vector w_{lj} of $\tilde{V}(\Lambda)$ with weight $\Lambda - \varepsilon_l + \delta j$, such that

$$E_i w_{lj} = 0 \qquad \forall i \neq l$$
$$E_l w_{lj} = (a_l + a_{l+1} + \ldots + a_m - a_{m+1} - \ldots - a_{m+j-1} + m - l - j + 1) w_{l+1,j} \qquad (3.22)$$

where $w_{l+1,j}$ is a weight vector with weight $\Lambda - \varepsilon_{l+1} + \delta_j$ for $l < m$.

Proof. From lemma 2 we see that (3.22) is true for $l = m$. Hence we will prove lemma 3 by induction on l. Thus, we may suppose that there exists a vector w_{lj} of the following form:

$$w_{lj} = \sum_{(k)} \lambda_k P_k(F_l, F_{l+1}, \ldots, F_{m+j-1}) v_\Lambda \qquad (3.23)$$

such that (3.22) holds. Herein, λ_k is a coefficient dependent on $a_{l+1}, a_{l+2}, \ldots, a_{m+j-1}$. We put

$$x = F_{l-1} w_{lj} = F_{l-1} \sum_k \lambda_k P_k(F_l, \ldots, F_{m+j-1}) v_\Lambda$$
$$y = \sum_k \lambda_k P_k(F_l, \ldots, F_{m+j-1}) F_{l-1} v_\Lambda. \qquad (3.24)$$

Then, for $i \neq l$, $l-1$, we find that

$$E_i x = F_{l-1} E_i w_{lj} = 0 \qquad (3.25)$$

and in a similar way as in the proof of lemma 2, this implies

$$E_i y = 0 \qquad \text{for } i \neq l, l-1. \qquad (3.26)$$

Next, we take $E_i = E_l$. Then the induction hypothesis leads to

$$E_l x = (a_l + a_{l+1} + \ldots + a_m - a_{m+1} - \ldots - a_{m+j-1} + m - l - j + 1) F_{l-1} w_{l+1,j}. \qquad (3.27)$$

Similarly as in (3.17)-(3.18), we see that $E_l y$ can be deduced from $E_l x$ by replacing a_l by $a_l - c_{l,l-1} = a_l + 1$. Hence

$$E_l y = (a_l + a_{l+1} + \ldots + a_m - a_{m+1} - \ldots - a_{m+j-1} + m - l - j + 2) F_{l-1} w_{l+1,j}. \qquad (3.28)$$

So, we define

$$w_{l-1,j} = (a_l + \ldots + a_m - a_{m+1} - \ldots - a_{m+j-1} + m - l - j + 2)x$$
$$- (a_l + \ldots + a_m - a_{m+1} - \ldots - a_{m+j-1} + m - l - j + 1)y. \qquad (3.29)$$

Then, also $E_l w_{l-1,j} = 0$, and since $E_{l-1} x = H_{l-1} w_{lj} = (a_{j-1} + 1) w_{lj}$ and $E_{l-1} y = a_{l-1} w_{lj}$, one obtains

$$E_{l-1} w_{l-1,j} = (a_{l-1} + a_l + \ldots + a_m - a_{m+1} - \ldots - a_{m+j-1} + m - l - j + 2) w_{lj}. \qquad (3.30)$$

This proves lemma 3.

Theorem 1. Let $\beta \in \bar{\Delta}_1^+ = \Delta_1^+$. Then $(\Lambda+\rho, \beta)=0$ if and only if there exists a weight vector $v_{\Lambda-\beta}$ of $\tilde{V}(\Lambda)$ with weight $\Lambda-\beta$ such that $v_{\Lambda-\beta}$ is an eigenvector of the Borel subalgebra B. Moreover, such a vector $v_{\Lambda-\beta}$ is unique (up to a non-zero factor).

Proof. If $(\Lambda+\rho, \beta)=0$ then (2.24) and the previous lemmas show the existence of a vector $v_{\Lambda-\beta}$ with the required properties. On the other hand, let $v_{\Lambda-\beta}$ be a weight vector, which is an eigenvector of B. Since $\beta \in \Delta_1^+$, we can put $\beta = \varepsilon_l - \delta_j$ $(1 \leq l \leq m, 1 \leq j \leq n)$. Then, the most general form of a weight vector in $\tilde{V}(\Lambda)$ with weight $\Lambda-\beta$ is equal to

$$v_{\Lambda-\beta} = \sum_k \mu_k P_k(F_l, F_{l+1}, \ldots, F_{m+j-1}) v_\Lambda \tag{3.31}$$

with μ_k arbitrary coefficients and (k) belonging to the subset S of permutations of $(l, l+1, \ldots, m+j-1)$, as described in the proof of lemma 2. Since $v_{\Lambda-\beta}$ is an eigenvector of B, we have to require that $E_i v_{\Lambda-\beta}=0$, for all i. But

$$E_i v_{\Lambda-\beta} = 0 \qquad \text{for } i = l+1, l+2, \ldots, m+j-1 \tag{3.32}$$

are precisely the conditions that determine the coefficients μ_k (up to a factor), as can be seen from the proof of lemmas 2 and 3. Hence, $\mu_k = c\lambda_k$ for all k and $v_{\Lambda-\beta} = cw_{lj}$. This shows that $v_{\Lambda-\beta}$ is unique and

$$E_l v_{\Lambda-\beta} = (\Lambda+\rho, \beta) v' \tag{3.33}$$

implies that $(\Lambda+\rho, \beta)=0$, since $v_{\Lambda-\beta}$ is an eigenvector of B.

Note that theorem 1 can partly be deduced from theorem 3 of Kac (1978). The main difference, however, is that we have given an explicit construction of the vector $v_{\Lambda-\beta}$.

4. The character formula for atypical representations

Let $w \in W$ be the elements of the Weyl group of $\mathrm{sl}(m/n)_{\bar{0}} = \mathrm{sl}(m) \oplus \mathrm{sl}(n) \oplus \mathbb{C}$. Then the character of a typical finite-dimensional irrep is given by (Kac 1978)

$$\mathrm{ch}\, V(\Lambda) = \frac{\prod_{\beta \in \Delta_1^+} [\exp(\beta/2) + \exp(-\beta/2)]}{\prod_{\alpha \in \Delta_0^+} [\exp(\alpha/2) - \exp(-\alpha/2)]} \sum_{w \in W} \varepsilon(w) \exp[w(\Lambda+\rho)]. \tag{4.1}$$

Here, $\exp(\alpha)$ is the notation for the formal exponential. Since $\Delta_1^+ = \{\varepsilon_i - \delta_j\ (i=1, \ldots, m; j=1, \ldots, n)\}$, we may denote this set by $\Delta_1^+ = \{\beta_1, \beta_2, \ldots, \beta_{mn}\}$. Also, since

$$\rho_1 = \tfrac{1}{2} \sum_{i=1}^{mn} \beta_i \tag{4.2}$$

the following equality holds:

$$\prod_{i=1}^{mn} [\exp(\beta_i/2) + \exp(-\beta_i/2)] = \exp(\rho_1) + \sum_{i=1}^{mn} \exp(\rho_1 - \beta_i) + \sum_{1 \leq i < j \leq mn} \exp(\rho_1 - \beta_i - \beta_j) + \ldots + \exp(-\rho_1). \tag{4.3}$$

On the right-hand side of (4.3) there appear $(mn+1)$ parts with, respectively, 1, mn, $mn(mn-1)/2$, $mn(mn-1)(mn-2)/3!, \ldots, 1$ term(s). The $(k+1)$ part in (4.3) is of the form

$$\sum_{1\leq i_1<i_2<\ldots<i_k\leq mn} \exp(\rho_1-\beta_{i_1}-\beta_{i_2}-\ldots-\beta_{i_k}). \tag{4.4}$$

Since $w(\rho_1)=\rho_1$ (see Kac 1978, proposition 1.7c), Δ_1^+ is invariant under W and W acts transitively on Δ_1^+, we find that

$$\sum \exp(\rho_1-\beta_{i_1}-\ldots-\beta_{i_k})=\sum \exp(\rho_1-w(\beta_{i_1})-\ldots-w(\beta_{i_k})) \tag{4.5}$$

for any element $w\in W$, where the summation in (4.5) is the same as in (4.4). Hence, (4.3) is W invariant. Then, combining (4.1)-(4.5), one obtains

$$\begin{aligned} \operatorname{ch} V(\Lambda) \prod_{\alpha\in\Delta_0^+} &[\exp(\alpha/2)-\exp(-\alpha/2)] \\ &= \sum_{w\in W} \varepsilon(w)\exp[w(\Lambda+\rho_0)]+\sum_{i=1}^{mn}\left(\sum_{w\in W}\varepsilon(w)\exp[w(\Lambda+\rho_0-\beta_i)]\right)+\ldots \\ &\quad+\sum_{1\leq i_1<\ldots<i_k\leq mn}\left(\sum_{w\in W}\varepsilon(w)\exp[w(\Lambda+\rho_0-\beta_{i_1}-\ldots-\beta_{i_k})]\right)+\ldots \\ &\quad+\sum_{w\in W}\varepsilon(w)\exp[w(\Lambda+\rho_0-2\rho_1)]. \end{aligned} \tag{4.6}$$

Expression (4.6) contains 2^{mn} non-zero $\mathrm{sl}(m)\oplus\mathrm{sl}(n)$ characteristics if and only if all $\Lambda-\beta_{i_1}-\ldots-\beta_{i_k}$ correspond to highest weights of $\mathrm{sl}(m)\oplus\mathrm{sl}(n)$, i.e., if and only if

$$\begin{aligned} &\text{all } a_i\geq n &&\text{for } 1\leq i\leq m-1 \\ &\text{all } a_i\geq m &&\text{for } m+1\leq i\leq m+n-1. \end{aligned} \tag{4.7}$$

In that case, it follows from (4.6) that the irrep $V(\Lambda)$ splits in 2^{mn} irreps of the Lie subalgebra $\mathrm{sl}(m)\oplus\mathrm{sl}(n)$.

Since $\mathrm{sl}(m/n)$ is a Lie superalgebra of type I, we have the following $\mathbb{Z}$ gradation:

$$\mathrm{sl}(m/n)=\mathrm{L}_{-1}\oplus\mathrm{L}_0\oplus\mathrm{L}_{+1} \tag{4.8}$$

where $\mathrm{L}_0=\mathrm{L}_{\bar{0}}$ and L_1 (resp L_{-1}) is the subalgebra of $\mathrm{sl}(m/n)$ spanned by the matrices $\left(\begin{smallmatrix}0&b\\0&0\end{smallmatrix}\right)$ (resp $\left(\begin{smallmatrix}0&0\\c&0\end{smallmatrix}\right)$) in (2.1). Put

$$\mathrm{P}=\mathrm{L}_0\oplus\mathrm{L}_{+1}. \tag{4.9}$$

Let Λ be a weight such that it is dominant integral for the Cartan subalgebra of $\mathrm{sl}(m)\oplus\mathrm{sl}(n)$. Then Λ determines unambiguously an $\mathrm{L}_{\bar{0}}$ module $V^0(\Lambda)$ which is irreducible and finite dimensional. We extend $V^0(\Lambda)$ to a P module by putting $\mathrm{L}_{+1}(V^0(\Lambda))=0$. Then we define, following Kac (1978), the induced L module

$$\bar{V}(\Lambda)=\mathrm{Ind}_{\mathrm{P}}^{\mathrm{L}}\, V^0(\Lambda). \tag{4.10}$$

Under these conditions, $\bar{V}(\Lambda)$ is finite dimensional and it contains a unique maximal submodule $\bar{I}(\Lambda)$ such that

$$V(\Lambda)=\bar{V}(\Lambda)/\bar{I}(\Lambda). \tag{4.11}$$

Now we make use of the following property (Kac 1978, proposition 2.1.c).

Proposition 1. Let $\mathrm{H}\subset\mathrm{L}$ be a subalgebra of L containing $\mathrm{L}_{\bar{0}}$ and $g_1,\ldots,g_t$ be odd elements of L whose projections onto L/H form a basis. Let Z be an H module. Then, one has the vector space decomposition

$$\mathrm{Ind}_{\mathrm{H}}^{\mathrm{L}}\mathrm{Z}=\bigoplus_{1\leq i_1<\ldots<i_s\leq t} g_{i_1}\ldots, g_{i_s}\mathrm{Z}. \tag{4.12}$$

We can apply this for H = P and Z = $V^0(\Lambda)$:

$$\bar{V}(\Lambda) = \bigoplus_{1\leq i_1<\ldots<i_s\leq mn} g_{i_1}\ldots g_{i_s}V^0(\Lambda) \tag{4.13}$$

where g_i is a root vector of the root space $L^{-\beta_i}$. The character of $V^0(\Lambda)$ is given by Weyl's character formula. Hence (4.13) implies

$$\text{ch}\ \bar{V}(\Lambda) = \prod_{i=1}^{mn}[1+\exp(-\beta_i)]\frac{\Sigma_{w\in W}\,\varepsilon(w)\exp[w(\Lambda+\rho_0)]}{\Pi_{\alpha\in\Delta_0^+}[\exp(\alpha/2)-\exp(-\alpha/2)]}. \tag{4.14}$$

Making use of arguments as in (4.4)-(4.5), this leads to

$$\text{ch}\ \bar{V}(\Lambda)\prod_{\alpha\in\Delta_{\bar{0}}^+}[\exp(\alpha/2)-\exp(-\alpha/2)]$$

$$= \sum_{1\leq i_1<\ldots<i_k\leq mm}\left(\sum_{w\in W}\varepsilon(w)\exp[w(\Lambda+\rho_0-\beta_{i_1}-\ldots-\beta_{i_k})]\right). \tag{4.15}$$

Note that the right-hand sides of (4.6) and (4.15) are equal, but (4.6) is valid only for typical weights Λ, whereas (4.15) can also be used for atypical weights (in both cases Λ is the dominant integral for $sl(m)\oplus sl(n)$). In fact, this shows that ch $V(\Lambda)$ = ch $\bar{V}(\Lambda)$ for Λ typical, which is obvious, since Kac (1978) proved that in this case $V(\Lambda)=\bar{V}(\Lambda)$. When Λ is an atypical highest weight, $\bar{V}(\Lambda)$ will not be irreducible, and then the question is: what is the maximal invariant subspace of $\bar{V}(\Lambda)$?

Obviously, $\bar{V}(\Lambda)$ is also an $L_{\bar{0}}$ module and, according to (4.15), it splits in 2^{mn} parts with highest weights given by $\Lambda-\beta_{i_1}-\ldots-\beta_{i_k}$ $(1\leq i_1<\ldots<i_k\leq mn)$. Let S be the set of sequences $s=(i_1,\ldots,i_k)$:

$$S=\{s=(i_1,\ldots,i_k)\,|\,1\leq i_1<\ldots<i_k\leq mn\}. \tag{4.16}$$

With every element s of S there corresponds a weight Λ_s under the mapping $\sigma: S\to H^*: s\to\Lambda_s=\Lambda-\beta_{i_1}-\ldots-\beta_{ik}$. The elements $\Lambda_s (s\in S)$ are called the weights of S. Note that S contains 2^{mn} elements, but to different elements of S there may correspond equal weights (for instance, if $\beta_i+\beta_j=\beta_k+\beta_l$). The weights of S are dominant (and hence correspond to highest weights of irreducible finite $L_{\bar{0}}$ modules) if (4.7) is satisfied. From now on, we shall suppose that this is the case. Let $v^0_{\Lambda_s} (s\in S)$ be the highest weight vectors of the $L_{\bar{0}}$ modules $V^0(\Lambda_s)$. Since these highest weight vectors are elements of $\bar{V}(\Lambda)$, they can be written in the following form:

$$v^0_{\Lambda_s}=\sum g_{j_1}\ldots g_{j_k}Q_{(j)}\otimes v_\Lambda \qquad s=(i_1,\ldots,i_k). \tag{4.17}$$

Here, the summation is over certain root vectors $g_{j_1},\ldots,g_{j_k}$ (corresponding to negative odd roots $-\beta_{j_1},\ldots,-\beta_{j_k}$) and elements $Q_{(j)}\in U(L_{\bar{0}})$ with weight $-\Sigma\, n_\alpha\alpha\,(\alpha\in\Delta_{\bar{0}}^+, n_\alpha\geq 0)$ such that

$$\beta_{i_1}+\ldots+\beta_{i_k}=\beta_{j_1}+\ldots+\beta_{j_k}+\Sigma\, n_\alpha\alpha \tag{4.18}$$

is satisfied. Note that due to (2.8) the number of odd roots on both sides of (4.18) is always the same. Also, (4.17) always contains a non-vanishing term of the form $g_{i_1}\ldots g_{i_k}\otimes v_\Lambda$.

Let us now consider the situation in which precisely *one* atypicality condition is fulfilled:

$$\begin{aligned}&(\Lambda+\rho,\beta_p)=0\\ &(\Lambda+\rho,\beta_i)\neq 0 \qquad \text{for } i\neq p.\end{aligned} \tag{4.19}$$

Then theorem 1 shows the existence of a B eigenvector $v^0_{\Lambda-\beta_p}$ of $\tilde{V}(\Lambda)$:

$$v^0_{\Lambda-\beta_p} = R \otimes v_\Lambda \qquad R \in \mathrm{U(L)}. \tag{4.20}$$

In fact, an explicit construction of R has been given in § 3. Note that R can be written in the following form:

$$R = g_p + g_{p'}T_{-\alpha_1} + g_{p''}T_{-\alpha_1'}T_{-\alpha_2'} + \ldots \tag{4.21}$$

where $\alpha_1, \alpha_1', \alpha_2', \ldots \in \Delta_{\bar{0}}^+$, T_α are the corresponding root vectors and

$$\beta_p = \beta_{p'} + \alpha_1 = \beta_{p''} + \alpha_1' + \alpha_2' = \ldots. \tag{4.22}$$

It is obvious that $v^0_{\Lambda-\beta_p}$ can be identified with an element of $\bar{V}(\Lambda)$, also given by (4.20):

$$v^0_{\Lambda-\beta_p} \in \bar{V}(\Lambda). \tag{4.23}$$

The weight $\Lambda-\beta_p$ determines an $\mathrm{L}_{\bar{0}}$ module $V^0(\Lambda-\beta_p)$ with highest weight vector $u_{\Lambda-\beta_p}$. Then, just as in (4.10), $\bar{V}(\Lambda-\beta_p)$ is well defined. The L module $\bar{V}(\Lambda-\beta_p)$ splits into a number of irreducible $\mathrm{L}_{\bar{0}}$ modules:

$$\bar{V}(\Lambda-\beta_p) = X_1 \oplus \ldots \oplus X_N \qquad N = 2^{mn}.$$

Let x_k be the $\mathrm{L}_{\bar{0}}$ highest weight vector of X_k. The weight of x_k is of the form $\sigma(s_k)-\beta_p$ for some $s_k \in S$. Suppose $s_k = (i_1, \ldots, i_l)$, then the analogue of (4.17) is

$$x_k = \sum g_{j_1} \ldots g_{j_l} Q'_{(j)} \otimes u_{\Lambda-\beta_p}. \tag{4.24}$$

We define a mapping A from the elements x_k in (4.24) to $\bar{V}(\Lambda)$ by replacing $u_{\Lambda-\beta_p}$ by $R \otimes v_\Lambda$ on the RHS of (4.24). Then A is extended to elements of the form qx_k, $q \in \mathrm{U}(\mathrm{L}_{\bar{0}})$ by putting $A(qx_k) = q(Ax_k)$. In this way, A is defined for all elements of $\bar{V}(\Lambda-\beta_p)$. Also, every element of $\bar{V}(\Lambda-\beta_p)$ can be uniquely written as a sum of elements of the form qx_k.

Lemma 4. The mapping $A: \bar{V}(\Lambda-\beta_p) \to \bar{V}(\Lambda)$ satisfies the following properties:

(a) it is a homomorphism of L modules: if $u, u' \in \bar{V}(\Lambda-\beta_p)$, $x \in \mathrm{L}$ and $x \cdot u = u'$, then $x \cdot Au = Au'$;

(b) x_k with weight $\sigma(s_k)-\beta_p$ is mapped into the zero vector if and only if $\sigma(s_k)-\beta_p \notin \sigma(S)$.

Proof. It is enough to prove (a) for elements of the form $u = qx_k$, $q \in \mathrm{U}(\mathrm{L}_{\bar{0}})$. But for such elements (a) follows from the fact that $v_{\Lambda-\beta_p}$ is a highest weight vector for L and the definitions of $\bar{V}(\Lambda-\beta_p)$ and $\bar{V}(\Lambda)$. As a consequence, A preserves the weight of weight vectors. To prove (b), let $s_k = (i_1, \ldots, i_l)$ and $\sigma(s_k)-\beta_p \notin \sigma(S)$. According to (a), Ax_k must be a highest weight vector for $\mathrm{L}_{\bar{0}}$. But all highest weight vectors for $\mathrm{L}_{\bar{0}}$ contained in $\bar{V}(\Lambda)$ are given by $v^0_{\Lambda_s}$. Hence $\sigma(s_k)-\beta_p \notin \sigma(S)$ implies $Ax_k = 0$. Conversely, if $\sigma(s_k)-\beta_p \in \sigma(S)$, then $p \notin \{i_1, \ldots, i_l\}$. Now x_k contains a unique term of the form $g_{i_1} \ldots g_{i_l} \otimes u_{\Lambda-\beta_p}$. Hence, according to (4.19) Ax_k contains a unique term of the form

$$g_{i_1} \ldots g_{i_l} g_p \otimes v_\Lambda$$

which shows that $Ax_k \neq 0$.

The image of $\bar{V}(\Lambda-\beta_p)$ under A, $\mathrm{Im}\,\bar{V}(\Lambda-\beta_p) = J(\Lambda-\beta_p)$, is a submodule of $\bar{V}(\Lambda)$ with highest weight $\Lambda-\beta_p$. From lemma 4, it follows that the $\mathrm{L}_{\bar{0}}$ components $V^0(\lambda)$

of $J(\Lambda-\beta_p)$ are those for which $\lambda\in\sigma(S)$ and for which λ is a highest weight of an $L_{\bar 0}$ module contained in $\bar V(\Lambda-\beta_p)$. Hence

$$\operatorname{ch} J(\Lambda-\beta_p)\prod_{\alpha\in\Delta_0^+}[\exp(\alpha/2)-\exp(-\alpha/2)]$$

$$=\sum_{\substack{1\le i_1<\ldots<i_k\le mn\\ i_1\ne p,\ldots,i_k\ne p}}\left(\sum_{w\in W}\varepsilon(w)\exp[w(\Lambda-\beta_p+\rho_0-\beta_{i_1}-\ldots-\beta_{i_k})]\right). \tag{4.25}$$

Any other submodule of $\bar V(\Lambda)$ must have a highest weight from S. But from theorem 1 it follows that no weight vector of $\bar V(\Lambda)$ with weight $\Lambda-\beta_i(i\ne p)$ can be a highest weight vector for L, and from Kac (1978, theorem 3b) one can conclude that no other weights of the form $\Lambda-\beta_{i_1}-\ldots-\beta_{i_k}$ $(i_1\ne p,\ldots,i_k\ne p)$ correspond to highest weight vectors for L. Hence

$$\bar I(\Lambda)=J(\Lambda-\beta_p). \tag{4.26}$$

Then $\operatorname{ch}V(\Lambda)=\operatorname{ch}\bar V(\Lambda)-\operatorname{ch}J(\Lambda-\beta_p)$, or explicitly

$$\operatorname{ch} V(\Lambda)\prod_{\alpha\in\Delta_0^+}[\exp(\alpha/2)-\exp(-\alpha/2)]$$

$$=\sum_{w\in W}\varepsilon(w)\exp[w(\Lambda+\rho_0)]+\sum_{\substack{i=1\\ i\ne p}}^{mn}\left(\sum_{w\in W}\varepsilon(w)\exp[w(\Lambda+\rho_0-\beta_i)]\right)+\ldots$$

$$+\sum_{\substack{1\le i_1<\ldots<i_k\le mn\\ i_1\ne p,\ldots,i_k\ne p}}\left(\sum_{w\in W}\varepsilon(w)\exp[w(\Lambda+\rho_0-\beta_{i_1}-\ldots-\beta_{i_k})]\right)+\ldots$$

$$+\sum_{w\in W}\varepsilon(w)\exp[w(\Lambda+\rho_0-2\rho_1+\beta_p)]. \tag{4.27}$$

We shall now obtain a more convenient form of (4.27). For this purpose, let w_1 be a fixed element of W and consider in all the parts of (4.27) the term with $w=w_1$. This is equal to

$$\varepsilon(w_0)\exp[w_1(\Lambda+\rho_0)]\left(1+\sum_{i\ne p}\exp[-w_1(\beta_i)]+\ldots+\exp[-w_1(2\rho_1-\beta_p)]\right). \tag{4.28}$$

The right-hand side is

$$\prod_{\substack{i=1\\ i\ne p}}^{mn}\{1+\exp[-w_1(\beta_i)]\}$$

$$=w_1\left(\prod_{i=1}^{mn}[1+\exp(-\beta_i)][1+\exp(-\beta_p)]^{-1}\right)$$

$$=w_1\left(\exp(-\rho_1+\beta_p/2)\prod_{i=1}^{mn}[\exp(\beta_i/2)+\exp(-\beta_i/2)]\right.$$

$$\left.\times[\exp(\beta_p/2)+\exp(-\beta_p/2)]^{-1}\right). \tag{4.29}$$

But $w_1(\rho_1)=\rho_1$ and $\Pi[\exp(\beta_i/2)+\exp(-\beta_i/2)]$ is W invariant. Hence, (4.29) equals

$$\exp[-\rho_1+w_1(\beta_p/2)]\frac{\Pi_{i=1}^{mn}[\exp(\beta_i/2)+\exp(-\beta_i/2)]}{\{\exp[w_1(\beta_p/2)]+\exp[-w_1(\beta_p/2)]\}} \tag{4.30}$$

Combining the previous results, we find the following.

Theorem 2. Let Λ be the highest weight of an atypical representation for which the conditions (4.7) and (4.19) are fulfilled. Then

$$\begin{aligned}\operatorname{ch} V(\Lambda) \prod_{\alpha\in\Delta_0^+}&[\exp(\alpha/2)-\exp(-\alpha/2)]\\ &=\sum_{i=1}^{mn}[\exp(\beta_1/2)+\exp(-\beta_1/2)]\ldots\widehat{[\exp(\beta_i/2)+\exp(-\beta_i/2)]}\ldots\\ &\quad\ldots[\exp(\beta_{mn}/2)+\exp(-\beta_{mn}/2)]\Big(\sum_{w\in W_{pi}}\varepsilon(w)\exp[w(\Lambda+\rho+\tfrac{1}{2}\beta_p)]\Big)\end{aligned} \tag{4.31}$$

where $W_{pi}=\{w\in W\,|\,w(\beta_p)=\beta_i\}$ and $\widehat{\ }$ is written above the factor to be deleted.

Note that (4.31) coincides with expression (2.11) of Sharp *et al* (1985) in the case of $\mathrm{sl}(m/n)$.

We have tried to find a similar expression for representations where more than one atypicality condition is fulfilled, but have not obtained a satisfactory result.

5. Example

In this section we shall give an example of an atypical representation of the Lie superalgebra $\mathrm{L}=\mathrm{sl}(3/2)$. In particular we shall consider a representation with highest weight Λ which is 'low lying' (this means for which (4.7) is not fulfilled). Although the proof of (4.31) only works for highest weight representations satisfying (4.7), the following example indicates that (4.31) may be true for all single-atypical representations.

The simple roots of sl(3/2) are, in the same notation as § 2,

$$\varepsilon_1-\varepsilon_2 \qquad \varepsilon_2-\varepsilon_3 \qquad \varepsilon_3-\delta_1 \qquad \delta_1-\delta_2. \tag{5.1}$$

For $\lambda\in\mathrm{H}^*$, the labels of λ are (a_1, a_2, a_3, a_4) with

$$a_i=\lambda(H_i). \tag{5.2}$$

Then one can express λ as follows:

$$\lambda=(a_1+a_2+a_3-a_4)\varepsilon_1+(a_2+a_3-a_4)\varepsilon_2+(a_3-a_4)\varepsilon_3+a_4\delta_1. \tag{5.3}$$

Note that the following equation holds in H^*:

$$(\varepsilon_1+\varepsilon_2+\varepsilon_3)-(\delta_1+\delta_2)=0. \tag{5.4}$$

The elements β_i of Δ_1^+ are determined by

$$(\beta_1,\beta_2,\ldots,\beta_6)=(\varepsilon_1-\delta_1, \varepsilon_1-\delta_2, \varepsilon_2-\delta_1, \varepsilon_2-\delta_2, \varepsilon_3-\delta_1, \varepsilon_3-\delta_2). \tag{5.5}$$

In this order, the six atypicality conditions are

$$\begin{aligned} &(1) \quad a_1+a_2+a_3+2=0 \\ &(2) \quad a_1+a_2+a_3-a_4+1=0 \\ &(3) \quad a_2+a_3+1=0 \\ &(4) \quad a_2+a_3-a_4=0 \\ &(5) \quad a_3=0 \\ &(6) \quad a_3-a_4-1=0. \end{aligned} \tag{5.6}$$

The even part of sl(3/2) is sl(3)⊕sl(2)⊕u(1). In general, sl(m/n) contains a u(1) generator h such that

$$[h, x]=jx \qquad \text{for } x \in \mathrm{L}_j, j \in \{-1, 0, +1\}$$

where $\mathrm{L}_{-1} \oplus \mathrm{L}_0 \oplus \mathrm{L}_{+1}$ is the previously mentioned $\mathbb{Z}$ gradation. An explicit expression for h is given by

$$h=\frac{n}{n-m}\sum_{i=1}^{m} iH_i-\frac{m}{n-m}\sum_{i=1}^{n-1}(n-i)H_{m+i}. \tag{5.7}$$

For sl(3/2) we find

$$h=-2(H_1+2H_2+3H_3)+3H_4. \tag{5.8}$$

Every dominant integral weight λ with labels (a_1, a_2, a_3, a_4) is also the highest weight of an irreducible $\mathrm{L}_{\bar{0}}$ module $V^0(\lambda)$. This module $V^0(\lambda)$ is characterised by the sl(3) Cartan-Dynkin labels (a_1, a_2), the sl(2) label (a_4) and the eigenvalue of the u(1) generator $k=-2(a_1+2a_2+3a_3)+3a_4$. Hence, $V^0(\lambda)$ is determined by the following set of labels:

$$(a_1, a_2/a_4/k). \tag{5.9}$$

Let Λ be the highest weight with labels $(1, 0, 0, 1)$. Clearly Λ satisfies exactly one of the atypicality conditions, namely (5.6(5)). Obviously, (4.7) is not fulfilled. However, after some lengthy calculations, we find that (4.31) gives rise to the following $\mathrm{L}_{\bar{0}}$ module structure of $V(\Lambda)$ (in the notation of (5.9)):

$$(1,0/1/1)\oplus(0,0/2/0)\oplus(0,0/0/0)\oplus(1,1/0/0)\oplus(0,1/1/-1). \tag{5.10}$$

Thus, (4.31) yields the correct character formula. Indeed, $(1, 0, 0, 1)$ is the adjoint representation of sl(3/2): $(1, 0/1/1)$ is $\mathrm{sl}_3 \otimes \mathrm{sl}_2^*$ (with u(1) value +1), $(0, 0/2/0)$ is the three-dimensional adjoint representation of sl(2), $(0, 0/0/0)$ represents u(1), $(1, 1/0/0)$ is the eight-dimensional sl(3) representation and $(0, 1/1/-1)$ is the six-dimensional sl(3)⊕sl(2) irrep $\mathrm{sl}_3^* \otimes \mathrm{sl}_2$ (with u(1)-value −1).

This example (and some other examples studied by the author) argues that, although the proof of (4.31) is not valid for 'low lying' representations, the final character formula of theorem 2 is true for *all* single-atypical representations.

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Note added in revision. It was pointed out by the referee that the definition of the mapping A after equation (4.24) is not completely convincing. For example, if $qx_k = q'x_k$ for $q \neq q'$, one has to show that this implies $qAx_k = q'Ax_k$. It would be natural to seek for an alternative definition of the mapping A where its properties are unambiguously clear but we have been unsuccessful in such an attempt.

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